THE PRIMES CONTAIN ARBITRARILY LONG ARITHMETIC PROGRESSIONS

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Abstract. We prove that there are arbitrarily long arithmetic progressions of primes.

There are three major ingredients.
[...]

[...]
for all $x \in \mathbb{Z}_N$ (here $(m_0, t_0, L_0) = (3, 2, 1)$) and

$$\begin{aligned}\mathbb{E}\Big(&\nu((x-y)/2)\nu((x-y+h_2)/2)\nu(-y)\nu(-y-h_1) \times \\ &\times \nu((x-y')/2)\nu((x-y'+h_2)/2)\nu(-y')\nu(-y'-h_1) \times \\ &\times \nu(x)\nu(x+h_1)\nu(x+h_2)\nu(x+h_1+h_2) \Big| x, h_1, h_2, y, y' \in \mathbb{Z}_N\Big) \\ &= 1 + o(1) \qquad (0.1)\end{aligned}$$

(here $(m_0, t_0, L_0) = (12, 5, 2)$).
[...]

Proposition 0.1 (Generalised von Neumann). *Suppose that ν is k-pseudorandom. Let $f_0, \ldots, f_{k-1} \in L^1(\mathbb{Z}_N)$ be functions which are pointwise bounded by $\nu + \nu_{\text{const}}$, or in other words*

$$|f_j(x)| \leqslant \nu(x) + 1 \text{ for all } x \in \mathbb{Z}_N, 0 \leqslant j \leqslant k-1. \qquad (0.2)$$

Let $c_0, \ldots, c_{k-1}$ be a permutation of $\{0, 1, \ldots, k-1\}$ (in practice we will take $c_j := j$). Then

$$\mathbb{E}\Big(\prod_{j=0}^{k-1} f_j(x + c_j r) \Big| x, r \in \mathbb{Z}_N\Big) = O\big(\inf_{0 \leqslant j \leqslant k-1} \|f_j\|_{U^{k-1}}\big) + o(1).$$

[...]

References

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[...]

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