

THE PRIMES CONTAIN ARBITRARILY LONG ARITHMETIC PROGRESSIONS

BEN GREEN AND TERENCE TAO

ABSTRACT. We prove that there are arbitrarily long arithmetic progressions of primes.

There are three major ingredients.

[...]

[...]

for all $x \in \mathbb{Z}_N$ (here $(m_0, t_0, L_0) = (3, 2, 1)$) and

$$\begin{aligned} & \mathbb{E} \left(\nu((x-y)/2) \nu((x-y+h_2)/2) \nu(-y) \nu(-y-h_1) \times \right. \\ & \quad \times \nu((x-y')/2) \nu((x-y'+h_2)/2) \nu(-y') \nu(-y'-h_1) \times \\ & \quad \left. \times \nu(x) \nu(x+h_1) \nu(x+h_2) \nu(x+h_1+h_2) \mid x, h_1, h_2, y, y' \in \mathbb{Z}_N \right) \\ & = 1 + o(1) \end{aligned} \tag{0.1}$$

(here $(m_0, t_0, L_0) = (12, 5, 2)$).

[...]

Proposition 0.1 (Generalised von Neumann). *Suppose that ν is k -pseudorandom. Let $f_0, \dots, f_{k-1} \in L^1(\mathbb{Z}_N)$ be functions which are pointwise bounded by $\nu + \nu_{\text{const}}$, or in other words*

$$|f_j(x)| \leq \nu(x) + 1 \text{ for all } x \in \mathbb{Z}_N, 0 \leq j \leq k-1. \tag{0.2}$$

Let c_0, \dots, c_{k-1} be a permutation of $\{0, 1, \dots, k-1\}$ (in practice we will take $c_j := j$). Then

$$\mathbb{E} \left(\prod_{j=0}^{k-1} f_j(x + c_j r) \mid x, r \in \mathbb{Z}_N \right) = O \left(\inf_{0 \leq j \leq k-1} \|f_j\|_{U^{k-1}} \right) + o(1).$$

[...]

REFERENCES

- [1] I. Assani, *Pointwise convergence of ergodic averages along cubes*, preprint.
- [2] A. Balog, *Linear equations in primes*, *Mathematika* **39** (1992) 367–378.
- [3] ———, *Six primes and an almost prime in four linear equations*, *Can. J. Math.* **50** (1998), 465–486.
- [4] V. Bergelson and A. Leibman, *Polynomial extensions of van der Waerden’s and Szemerédi’s theorems*, *J. Amer. Math. Soc.* **9** (1996), 725–753.

[...]

1991 *Mathematics Subject Classification*. 11N13, 11B25, 374A5.

While this work was carried out the first author was a PIMS postdoctoral fellow at the University of British Columbia, Vancouver, Canada. The second author was a Clay Prize Fellow and was supported by a grant from the Packard Foundation.

SCHOOL OF MATHEMATICS, UNIVERSITY WALK, BRISTOL, BS8 1TW

E-mail address: `b.j.green@bristol.ac.uk`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT LOS ANGELES, LOS ANGELES
CA 90095

E-mail address: `tao@math.ucla.edu`